CMS Technical Summary Report #88-8

UNIVERSITY OF WISCONSIN

AD-A193 478

A MINIMAX METHOD
FOR A CLASS OF HAMILTONIAN SYSTEMS
WITH SINGULAR POTENTIALS
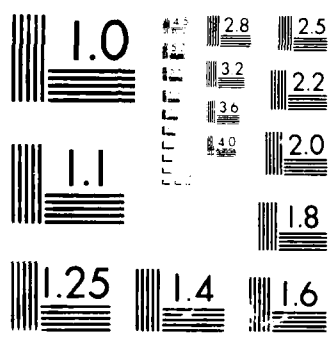
Abbas Bahri
and
Paul H. Rabinowitz
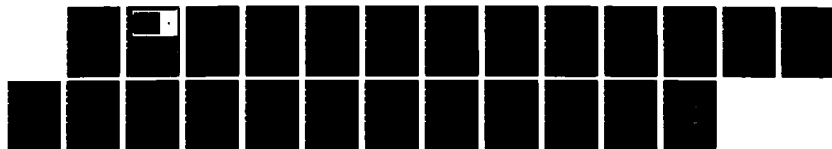
CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

October 1987

(Received August 25, 1987)

88 3 23 053

UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

A MINIMAX METHOD FOR A CLASS OF HAMILTONIAN
SYSTEMS WITH SINGULAR POTENTIALS

Abbas Bahri[*] and Paul H. Rabinowitz[**]

CMS Technical Summary Report #88-8

October 1987

ABSTRACT

A-1

This paper presents a minimax method which gives existence and

multiplicity results for time periodic solutions of a class of Hamiltonian

systems when a singular potential is present. The singularity satisfies the

strong force condition of Gordon. When milder singularities are permitted a

notion of generalized T-periodic solution is introduced and we get existence

and multiplicity results for such solutions.

AMS (MOS) Subject Classifications: 34C25, 58E05, 70F99

Key Words: minimax method, calculus of variations, Hamiltonian systems,

singular potential, strong force condition, collision orbits,

weak force condition, generalized T-periodic solution

A MINIMAX METHOD FOR A CLASS OF HAMILTONIAN

[*]
  Department of Mathematics, Ecole Polytechnique, Palaiseau, France.
[**]
  Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706.

A MINIMAX METHOD FOR A CLASS OF HAMILTONIAN
SYSTEMS WITH SINGULAR POTENTIALS

Abbas Bahri[*] and Paul H. Rabinowitz[**]

Introduction

Recently there has been a considerable amount of work on the existence of time periodic solutions of prescribed period for Hamiltonian systems of the form

$$\text{(HS)} \qquad \ddot{q} + V'(q) = 0 \ .$$

Here $q = (q_1, \ldots, q_n)$, $n > 2$, $V : \mathbf{R}^n \backslash S \to \mathbf{R}$, V and $V' \to 0$ as $|q| \to \infty$ and V is singular on S, i.e. $|V(q)| \to \infty$ as $q \to S$. The case where V depends explicitly on t in a T-periodic fashion has also been treated. See e.g. Ambrosetti and Coti-Zelati [1-2], Coti-Zelati [3], Degiovanni, Giannoni, and Marino [4], Greco [5-6], and especially the extensive bibliographies of [1] and [3]. These papers were motivated in part by earlier work of Gordon [7] which is mainly for $n = 2$. Our own study of singular Hamiltonian systems was a consequence of our interest in [5]; we only learned later of [1-4].

The major focus of this paper is with singular potentials for which S is a single point which is taken to be the origin. Slight modifications of our methods permit us to treat more general compact sets S. To describe our results assume V satisfies:

(V_1) $V \in C^1(\mathbf{R}^n \backslash \{0\}, \mathbf{R})$, $n > 3$,

[*] Department of Mathematics, Ecole Polytechnique, Palaiseau, France.
[**] Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706.

(V_2) $V(q) < 0$ and $V(q)$, $V'(q) \to 0$ uniformly as $|q| \to \infty$,

(V_3) $-V(q) \to \infty$ as $q \to 0$,

(V_4) There is a neighborhood W, of 0 in $\mathbb{R}^n$ and a function

   $U \in C^1(W\setminus\{0\}, \mathbb{R})$ such that $U(q) \to \infty$ as $q \to 0$ and $-V(q) > |U'(q)|^2$

   for $q \in W\setminus\{0\}$,

(V_5) $K \equiv \{V(q) \mid V'(q) = 0\}$ is bounded.

Hypothesis (V_4) governs the rate at which $-V(q) \to \infty$ as $q \to 0$. It was

introduced by Gordon who called it the strong force condition. If e.g.

$V(q) = -|q|^{-\beta}$ for q near 0, (V_4) is satisfied if $\beta > 2$. Thus the

Coulomb potential with $\beta = 2$ satisfies (V_4) but the gravitational potential

with $\beta = 1$ is excluded.

   We have two types of results for (HS). First in §1 under hypotheses like

$(V_1) - (V_4)$, the existence of T periodic solutions of (HS) for any $T > 0$

will be established. Other authors [1-3], [5-6] have obtained similar

results. Like them, we will use the calculus of variations to obtain

solutions of (HS) as critical points of the corresponding functional

(0.1) $$I(q) \equiv \int_0^T [\tfrac{1}{2} |\dot{q}|^2 - V(q)]dt \ .$$

The main novelty in our treatment of I is our rather geometrical minimax

characterization of a corresponding critical value, c. This approach leads

to an estimate in §2 relating c and T of the form

(0.2) $$T < c\varphi(2(2Tc)^{1/2})$$

where

$$\varphi(r) = \max_{0<|q|<r} -(V(q))^{-1} \ .$$

Inequality (0.2) leads to multiplicity results for (HS) like:

<u>Theorem 0.3</u>: If V satisfies $(V_1) - (V_5)$, then for any $T > 0$, (HS) possesses infinitely many distinct nonconstant T-periodic solutions.

Our second type of result for (HS) is also based on inequality (0.2) and involves removing the condition (V_4). One of the consequences of (V_4) is that if $q \in W^{1,2}$ and V satisfies (V_1), $(V_3) - (V_4)$ then $q(t) \neq 0$ for all $t \in [0,T]$. However if (V_4) is eliminated, it is possible that a $W^{1,2}$ solution q of (HS) can vanish somewhere, i.e. enter the singularity of V. Such "collision" orbits cannot be classical solutions of (HS). Thus a broader notion of solution is needed. See also [8] in this regard. In §3 a notion of generalized T-periodic solutions of (HS) will be introduced. With the aid of the existence results of §1 - 2, and an approximation argument, it will be shown that (HS) possesses multiple generalized T-periodic solutions. E.g. we have

<u>Theorem 0.4</u>: If V satisfies $(V_1) - (V_3)$, then for each $T > 0$, (HS) has infinitely many distinct generalized T-periodic solutions

§1. A minimax method to solve (HS)

In order to establish the existence of periodic solutions of (HS) via a minimax argument, a few preliminaries are required. Let $C_T(\mathbf{R}, \mathbf{R}^n)$ denote the Banach space of T-periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the usual L^∞ norm. Let $E_T \equiv W_T^{1,2}(\mathbf{R}, \mathbf{R}^n)$ denote the space of T-periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\|q\| = (\int_0^T |\dot{q}|^2 dt + [q]^2)^{1/2}$$

where

$$[q] = \frac{1}{T} \int_0^T q(t) dt \quad .$$

When there is no ambiguity, the subscript T will be omitted. Note that

$E_T \subset C_T(\mathbf{R}, \mathbf{R}^n)$.   Let

$$\Lambda = \{q \in E_T \mid q(t) \neq 0 \text{ for all } t \in [0,T]\} \quad ,$$

i.e. Λ is the subset of E of loops which avoid the origin. Clearly Λ is

an open subset of E. Standard results and the definition of Λ imply that

if

(1.1)
$$I(q) = \int_0^T [\tfrac{1}{2} |\dot{q}|^2 - V(q)] dt \quad ,$$

then $I \in C^1(\Lambda, \mathbf{R})$ and any critical point of I on Λ is a classical

solution of (HS) [9]. The family of constant loops, $\mathbf{R}^n \backslash \{0\}$ lies in Λ and

will be denoted by $\tilde{\Lambda}$.

Critical points of I will be obtained by minimaxing I over certain

surfaces that will be introduced next. Let D^n denote the unit ball in $\mathbf{R}^n$.

For $n > 3$, let

$$\Gamma = \{h \in C(D^{n-2}, \Lambda) \mid h\big|_{S^{n-3}} \equiv \text{constant}\}$$

while for $n = 3$

$$\Gamma = \{h \in C([-1,1], \Lambda) \mid h(1) \text{ and } h(-1) \in \tilde{\Lambda}\} \quad .$$

Identifying 0 and T, and the interval $[0,T]$ with S^1, associated with

each $h \in \Gamma$ is a map $\tilde{h} \in C(D^{n-2} \times S^1, S^{n-1})$ defined by

$$\tilde{h}(x,t) = \frac{h(x)(t)}{|h(x)(t)|} \quad .$$

For $n > 3$, this map is constant on $S^{n-3} \times S^1$, the boundary of $D^{n-2} \times S^1$.

Therefore $\tilde{h}$ can be considered to be a map from $S^{n-2} \times S^1$ into S^{n-1}. As

such it has a degree which will be denoted by $\deg \tilde{h}$. The situation is a bit

different for $n = 3$. Then

$$\tilde{h} : [-1,1] \times S^1 \to S^2$$

with $\tilde{h}(1,t) \equiv \tilde{h}(1,T)$ and $\tilde{h}(-1,t) \equiv \tilde{h}(-1,T)$. Collapsing $\{-1\} \times S^1$ to a

-4-

point ξ and $\{1\} \times S^1$ to $-\xi$, $\tilde{h}$ may be viewed as a map of S^2 into S^2 and therefore has a degree, $\deg \tilde{h}$.

Let

$$\Gamma^* = \{h \in \Gamma \mid \deg \tilde{h} \neq 0\} \quad .$$

Lemma 1.2: $\Gamma^* \neq \emptyset$.

Proof: This is obvious if $n = 3$. If $n > 3$, more care is needed. The set $S^{n-2} \times S^1$ can be parametrized by (x, e^{it}) where $x \in \mathbf{R}^{n-1}$ with $|x|^2 = x_1^2 + \cdots + x_{n-1}^2 = 1$ and $t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Define a map $g : S^{n-2} \times S^1 \to S^{n-1}$ as follows:

$$g(x, e^{it}) = \left(x \, \overline{\sqrt{1 - \frac{4t^2}{\pi^2}}} \, , \frac{2t}{\pi}\right) \quad , \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$= (-\cos t, 0, \ldots, 0, \sin t) \quad , \quad t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \quad .$$

Then $g \in C(S^{n-2} \times S^1, S^{n-1})$. To calculate its degree, choose a regular point of g, e.g. $(-1, 0, \ldots, 0)$. The inverse image of this point is $(-1, 0, \ldots, 0) \times \{1\} \in S^{n-1} \times S^1$. Therefore $\deg g = 1$ or -1 depending on the orientation chosen for $S^{n-2} \times S^1$ and S^{n-1}. It is then easy to see that there exists $h \in \Gamma$ such that $g = \tilde{h}$. Therefore $\Gamma^* \neq \emptyset$.

Now a minimax value of I can be defined as

(1.3)
$$c = \inf_{h \in \Gamma^*} \max_{x \in D^{n-2}} I(h(x)) \quad .$$

We will show that under appropriate conditions on V, c is a positive critical value of I. A few technical points are required to do this. First:

Proposition 1.4: If V satisfies $(V_1) - (V_2)$ and

(V_6) There is an $\alpha > 0$ such that

-5-

$$\liminf_{q \to 0} -V(q) > \alpha \ ,$$

then $c > 0$.

Proof: If not, there is a sequence $(h_m) \subset \Gamma^*$ such that $I(h_m(x)) \to 0$ as $m \to \infty$ for all $x \in D^{n-2}$. By (1.1) and (V_2), as $m \to \infty$,

(1.5)
$$\|\frac{d}{dt} h_m(x)\|_{L^2} \to 0 \ .$$

Since for all $q \in E$,

(1.6)
$$\|q - [q]\|_{L^\infty} \leq T^{1/2} \|\dot{q}\|_{L^2} \ ,$$

(1.5) shows that

(1.7)
$$\|h_m(x) - [h_m(x)]\|_{L^\infty} \to 0$$

as $m \to \infty$. By (V_2) and (1.1) again,

(1.8)
$$\int_0^T V(h_m(x)(t))dt \to 0$$

as $m \to \infty$. Consequently by (1.7), (V_2), and (V_6), $[h_m(x)] \to \infty$. It follows that for large m, $h_m(\cdot)$ is homotopic in Γ to $[h_m(\cdot)]$. Therefore for $n > 3$ and large m, $\tilde{h}_m$ is homotopic to a map $\varphi \in C(S^{n-2} \times S^1, S^{n-1})$:

(1.9)
$$\varphi(x,t) = \frac{[h_m(x)]}{|[h_m(x)]|}$$

while for $n = 3$, $\varphi \in C(S^2, S^2)$ and is defined as in (1.9). For $n > 3$, φ factors through the projection map $S^{n-2} \times S^1 \to S^{n-2}$ since it is independent of t. Therefore $\deg \varphi = 0$. Similarly if $n = 3$, φ factors through the map $(x,t) \to t$ and $\deg \varphi = 0$. Thus in both cases for large m, $h_m \notin \Gamma^*$, a contradiction. Hence $c > 0$.

Remark 1.10: Suppose $S \supset \mathbf{R}^n$ is compact with $0 \in S$ and (V_1) is replaced by

(V_1') $V \in C^1(\mathbf{R}^n \backslash S, \mathbf{R})$, $n > 3$.

Let

$$\Lambda_S = \{q \in E \mid q(t) \notin S \text{ for all } t \in [0,T]\} \quad,$$

the set of loops in E which avoid S and let

$$\widetilde{\Lambda}_S = \{\xi \in \mathbf{R}^n \mid \xi \notin S\} \quad.$$

Finally set

$$\Gamma_S = \{h \in C(D^{n-2}, \Lambda_S) \mid h\big|_{S^{n-3}} \equiv \text{constant}\} \quad.$$

Then the proof of Lemma 1.4 yields: If V satisfies (V_1'), (V_2), and

(V_6') $\liminf\limits_{q \to S} - V(q) \geqslant \alpha > 0 \quad,$

then $c > 0$.

Next it will be shown that I satisfies the $(PS)^+$ condition on Λ, i.e.

$(PS)^+$: For any $s > 0$, if $(q_m) \subset \Lambda$, $I(q_m) \to s$ and $I'(q_m) \to 0$, then

q_m possesses a subsequence converging to some $q \in \Lambda$.

Note that by taking a sequence $q_m \in \widetilde{\Lambda}$ with $q_m \to \infty$, $I(q_m) \to 0$. Thus I does not satisfy (PS) at level 0 and does not attain its infimum. Indeed this is one of the difficulties in treating I variationally. Our argument is a variant of that of Greco [5]. It is easy to check that:

Lemma 1.11: Let V satisfy (V_1), $(V_3) - (V_4)$. If $(q_m) \subset \Lambda$ and q_m converges weakly in E and strongly in L^∞ to $q \in \partial\Lambda$, then

$$- \int_0^T V(q_m(t))dt \to \infty$$

(and therefore $I(q_m) \to \infty$).

Proof: See [5, Lemma 2.1].

Proposition 1.12: If V satisfies $(V_1) - (V_4)$, I satisfies $(PS)^+$ on Λ.

Proof: Let $s > 0$ and $(q_m) \subset \Lambda$ with $I(q_m) \to s$ and $I'(q_m) \to 0$. By (1.1), $(\dot{q}_m)$ is bounded in L^2. We claim $([q_m])$ is also bounded in $\mathbf{R}^n$ and

-7-

therefore (q_m) is bounded in E. If not, $|[q_m]| \to \infty$ along a sub-sequence. Hence by (1.6) and (V_2),

$$(1.13) \qquad\qquad \int_0^T V(q_m(t))dt \to 0$$

and

$$(1.14) \qquad\qquad \int_0^T V'(q_m) \cdot (q_m - [q_m])dt \to 0$$

as $m \to \infty$. Therefore by (1.13) - (1.14),

$$I(q_m) = \frac{1}{2} I'(q_m)(q_m - [q_m]) + \frac{1}{2} \int_0^T V'(q_m) \cdot (q_m - [q_m])dt$$

$$(1.15)$$

$$- \int_0^T V(q_m(t))dt \to 0 \quad .$$

But $I(q_m) \to s$, strictly positive, a contradiction. Now the boundedness of q_m in E and standard embedding theorems imply along a subsequence q_m converges weakly in E and strongly in L^∞ to $q \in E$. Since $I(q_m) \to s$, Lemma 1.11 shows $q \in \Lambda$. Hence the form of I' shows $q_m \to q$ in E.

Remark 1.16: In the spirit of Remark 1.10, if (V_1) is replaced by (V_1') and (V_3) - (V_4) by

(V_3') $\lim\limits_{q \to S} - V(q) = \infty$,

(V_4') There is a neighborhood W of S in $\mathbb{R}^n$ and $U \in C^1(W\backslash S, \mathbb{R})$ such that $U(q) \to \infty$ as $q \to S$ and $-V(q) > |U'(q)|^2$ for $q \in W\backslash S$,

then the above proof shows $(PS)^+$ holds for Λ_S.

The next step in showing that c as given by (1.3) is a critical value of I is a version of a standard "Deformation Theorem" that is appropriate for our setting. For $\sigma \in \mathbb{R}^+$, let $A_\sigma = \{q \in \Lambda \mid I(q) \leq \sigma\}$ and $K_\sigma = \{q \in \Lambda \mid I(q) = \sigma$ and $I'(q) = 0\}$.

Proposition 1.17: Let V satisfy (V_1) - (V_4). Suppose s is not a critical value of I. Then for all $\bar{\epsilon} > 0$ there is an $\epsilon > 0$ and $\eta \in C([0,1] \times \Lambda, \Lambda)$ such that

-8-

$1°$ $\eta(1,q) = q$ if $I(q) \notin (s-\bar{\epsilon}, s+\bar{\epsilon})$,

$2°$ $I(\eta(s,q)) \leq I(q)$ for $s \geq 0$

$3°$ $\eta(1,A_{s+\epsilon}) \subset A_{s-\epsilon}$,

$4°$ $\eta(1,\cdot) : \tilde{\Lambda} \to \tilde{\Lambda}$,

$5°$ $\eta(1,\cdot) : \Gamma^* \to \Gamma^*$.

Proof: The proof of a "standard" version of the Deformation Theorem can be found in [9, Appendix A]. We will only indicate the minor modifications in its proof needed to handle the differences in structure encountered here. The map η is a solution of a differential equation of the form

$$(1.18) \qquad \frac{d\eta}{dt} = \omega(\eta)\Psi(\eta) \quad , \quad \eta(0,q) = q$$

where $0 \leq \omega \leq 1$ is a cut-off function and Ψ is a pseudogradient vector field for I'. The choice of ω guarantees that $1°$ holds and the choice of Ψ gives $2°$. Since $I(q) \to \infty$ as $q \to \partial\Lambda$, $2°$ and Lemma 1.11 show that $\eta \in C([0,1] \times \Lambda, \Lambda)$.

Property $3°$ follows as in [9]. Note that I is invariant under a natural S^1 symmetry, namely

$$(1.19) \qquad\qquad I(q(t+\theta)) = I(q(t))$$

for all $\theta \in \mathbb{R}/[0,T]$. Hence w can be chosen so that it is also invariant under the S^1 action and Ψ so that it is equivariant with respect to this action. The fixed point set of the action in Λ is $\tilde{\Lambda}$. Hence $\eta(t,\cdot) : \tilde{\Lambda} \to \tilde{\Lambda}$ and $\eta(t,\cdot)$ is also S^1 equivariant. Finally if $h \in \Gamma^*$, $h_s \equiv \eta(s,h) \in \Gamma$ via $4°$ and the associated map $\tilde{h}_s$ from $S^2 \to S^2$ if $n = 3$ or from $S^{n-2} \times S^1$ to S^{n-1} for $n > 3$ is clearly homotopic to $\tilde{h}$. Therefore $\deg \tilde{h}_s = \deg \tilde{h} \neq 0$ and $5°$ holds.

Theorem 1.20: If V satisfies $(V_1) - (V_4)$, then for each $T > 0$, I has a critical value c given by (1.3) with a corresponding critical point $q \in \Lambda$

-9-

which is a classical solution of (HS).

Proof: Since (V_3) holds, $c > 0$ via Proposition 1.4. Using 3° and 4° in a standard way - see e.g. [9] - shows that c is a critical value of I in Λ. Lastly a simple regularity argument shows q is a classical solution of (HS).

Remark 1.21: If V satisfies (V_1'), (V_2), (V_3'), and (V_4'), Remark 1.10 and 1.16 and a slightly modified proof of Theorem 1.20 shows the conclusions of Theorem 1.20 also hold for this setting. See also [1].

Remark 1.22: Since Theorem 1.20 holds for all $T > 0$, if (HS) has no equilibrium solutions, a sequence of T periodic solutions of (HS), (distinct in E_T), can be constructed as follows: For each $k \in \mathbf{N}$, let $q_k(t)$ be a solution of (HS) of period T/k given by Theorem 1.20. Say $q_1(t)$ has minimal period $T/_{k_1}$. Then for $k > k_1$, $q_k(t)$ is distinct from $q_1(t)$. Similar reasoning shows that infinitely many of the functions $q_k(t)$ are distinct. However if (HS) has equilibrium solutions, a more careful argument is needed to get multiple solutions and thus will be carried out in §2.

§2. A lower bound for c and its consequences

In this section, the lower bound (0.2) for c will be derived and it will be used to get better existence and multiplicity results for (HS) than in §1.

Proposition 2.1: Let V satisfy (V_1) - (V_3) and let c be as in (1.3). If $\varphi(0) = 0$ and for $r > 0$,

$$(2.2) \qquad \varphi(r) = \max_{0 < |q| < r} - (V(q))^{-1} ,$$

then

-10-

(2.3)
$$T < c_\varphi(2(2Tc)^{1/2}) \quad .$$

Proof: $(V_1) - (V_3)$ imply that φ is a continuous monotone nondecreasing function with $\varphi(r) \to \infty$ as $r \to \infty$. For $\theta > 0$, set

(2.4)
$$\varphi_\theta(r) = \theta r + \varphi(r) \quad .$$

Then φ_θ has the same properties as φ and is strictly monotone increasing. Let $q \in \Lambda$. It can be written as

(2.5)
$$q(t) = \xi + Q(t)$$

where $\xi = [q]$ and Q is orthogonal in E to $\mathbb{R}^n \subset E$. Suppose

(2.6)
$$I(q) < b \quad .$$

Then (1.1), (1.6), and (2.6) imply

(2.7)
$$\|Q\|_{L^\infty} < (2bT)^{1/2} \quad .$$

Now by (V_2), (1.1), and (2.4),

$$T = \int_0^T \left(\frac{-V(q)}{-V(q)}\right)^{1/2} dt < (\int_0^T - V(q)dt)^{1/2} (\int_0^T - (V(q))^{-1} dt)^{1/2}$$

(2.8)
$$< b^{1/2} \int_0^T \varphi_\theta(|q|)dt \quad .$$

Substituting (2.5) into (2.8) and using (2.7) and the monotonicity of φ_θ then shows

(2.9)
$$T^2/b < T \varphi_\theta(|\xi| + \|Q\|_{L^\infty}) < T \varphi_\theta(|\xi| + (2bT)^{1/2}) \quad .$$

Consequently

(2.10)
$$\varphi_\theta^{-1}(T/b) - (2bT)^{1/2} < |\xi| \quad .$$

Let $s \in [0,1]$. Then by (2.10) and (2.7),

(2.11)
$$|\xi + sQ(t)| > |\xi| - s\|Q\|_{L^\infty} > \varphi_\theta^{-1}(T/b) - 2(2bT)^{1/2} \quad .$$

Suppose that

(2.12)
$$\varphi_\theta^{-1}(T/b) - 2(2bT)^{1/2} > 0 \quad .$$

Then (2.11) - (2.12) show those $q \in \Lambda$ satisfying (2.6) are homotopic in Λ to their mean values. If (2.12) held for $b = c+\varepsilon$ for some $\varepsilon > 0$, then

-11-

there would be an $h \in \Gamma^*$ such that $h(x)$ satisfies (2.6) for all $x \in D^{n-2}$.

Consequently the map $h(x)$ would be homotopic to its mean value for all

$x \in D^{n-2}$ and therefore $\deg \tilde{h} = 0$, contrary to $h \in \Gamma^*$. Hence (2.12) cannot

hold for $b = c+\varepsilon$. Since this is the case for all $\varepsilon > 0$, it follows that

(2.13)
$$T < \varphi_\theta(2(2cT)^{1/2})$$

and letting $\theta \to 0$ yields (2.3).

Remark 2.14: (i) Suppose that V also depends on t in a T-periodic

fashion. Then the above argument goes through virtually unchanged to yield

(2.3) for this case.

(ii) Suppose that (V_1) and (V_3) are replaced by (V_1') and (V_3') (with $0 \in S$)

and the definition of φ is replaced by $\varphi(0) = 0$ and for $r > 0$,

$$\varphi(r) = \sup_{|q| < r, q \notin S} - (V(q))^{-1} .$$

If further Λ and Γ^* are replaced by Λ_S and Γ_S^*, it is easy to see that

the argument of Proposition 2.1 carries over unchanged yielding (2.3) for this

situation.

Two kinds of applications of (2.3) will be given. The first is to forced

versions of (HS):

(2.15)
$$\ddot{q} + V_q(t,q) = 0 .$$

For this setting we interpret $(V_1) - (V_4)$ to mean the natural extension of

these hypotheses to reflect the further dependence of V on t in a T-

periodic fashion.

Theorem 2.16: Suppose V is T-periodic in t and satisfies $(V_1) - (V_4)$.

Then (2.15) possesses a T-periodic solution.

Proof: Let c be defined as in (1.3). By Remark 2.14(i), the estimate (2.3)

holds. If c_1 is defined as a minimax of I over a subclass of Γ^*, then

$c_1 > c$. Let

$$\hat{\Gamma} = \{h \in \Gamma^* \mid I(h(\pm 1)) < \frac{c}{2}\}$$

if $n = 3$ or

$$\hat{\Gamma} = \{h \in \Gamma^* \mid I(h(x)) < \frac{c}{2} \text{ for } x \in S^{n-3}\}$$

if $n > 3$. Proposition 1.17 no longer holds in its entirety since V depends on t so I is no longer S^1 invariant. However the proof holds up to (1.19). Since $c_1 > c > \frac{c}{2}$, choosing $\overline{\epsilon} = c/3$, by $1°$ of Proposition 1.17, it follows that $\eta(1, \cdot) : \hat{\Gamma} \to \hat{\Gamma}$. Hence the reasoning of the proof of Theorem 1.20 shows c_1 is a critical value of I and the corresponding critical point is a solution of (2.15).

Combining the proof of Theorem 2.16 with Remarks 1.10, 1.16, and 2.14(ii) immediately gives:

<u>Theorem 2.17</u>: If V is T-periodic in t and satisfies (V_1'), (V_2), (V_3') – (V_4'), then (2.15) possesses a T-periodic solution.

As a second application of (2.3), some multiplicity results will be obtained in the setting of Theorem 1.20.

<u>Theorem 2.18</u>: If V satisfies $(V_1) - (V_4)$, I possesses an unbounded sequence of critical values.

Proof: For each $k \in \mathbb{N}$, Theorem 1.20 can be applied with T replaced by T/k to the functional

(2.19) $$I_k(q) = \int_0^{T/k} [\frac{1}{2} |\dot{q}|^2 - V(q)] dt$$

obtaining a critical value b_k and critical point $q_k \in W_{T/k}^{1,2} \cap \Lambda$. By (2.3),

(2.20) $$T/k < b_k \; \varphi(2(2Tk^{-1}b_k)^{1/2}) \; .$$

If $k \, b_k$ is bounded along some subsequence, (2.20) shows

(2.21) $$T < 0$$

-13-

which is impossible. Hence $k \, b_k \to \infty$ as $k \to \infty$. Considering q_k as an element of $W_T^{1,2}$, we have

(2.22)
$$c_k \equiv I(q_k) = k \, b_k \to \infty$$

as $k \to \infty$ and the Theorem follows.

Corollary 2.23: Under the hypotheses of Theorem 2.18, if further (V_5) holds, (HS) possesses infinitely many distinct nonconstant T-periodic solutions.

Proof: Note that for $q \in K$, $I(q) = -TV(q)$. By (V_5), I is bounded on K. Hence the result follows.

Corollary 2.24: Along a subsequence of $k \to \infty$, either

(2.25)
$$\min_{t \in [0,T]} |q_k(t)| \to 0$$

or

(2.26)
$$\|q_k\|_{L^\infty} \to \infty \quad .$$

Proof: By (2.22), $I(q_k) \to \infty$ as $k \to \infty$. The form of I and (V_1)-(V_4) imply either (2.24) holds or

(2.27)
$$\|\dot{q}_k\|_{L^2} \to \infty$$

along some subsequence of $k \to \infty$. (HS) implies

(2.28)
$$\|\dot{q}_k\|^2 = \int_0^T V'(q_k) \cdot q_k \, dt$$

so if neither (2.25) nor (2.26) were valid, by (2.28) and (V_1)-(V_4), $\dot{q}_k$ would be bounded in L^2, contrary to (2.27).

Remark 2.29: Corollary 2.24 implies infinitely many of the loops $q_k(t)$ are geometrically distinct.

Once again by applying the reasoning of Theorem 2.18 together with Remarks 1.10, 1.16, and (2.14)(ii), we get an analogous result where there is a more general singular set.

Theorem 2.30: If V satisfies (V_1'), (V_2), (V_3') - (V_4'), then for each T > 0, (HS) possesses infinitely many distinct T-periodic solutions. If further (V_5) holds, (HS) possesses infinitely many distinct nonconstant solutions.

Remark 2.31: Ambrosetti and Coti-Zelati [1, 3] have obtained analogues of Theorems 2.18 and 2.30 when V depends on t in a T-periodic fashion. Their arguments are less direct than ours. It is unclear as to whether they extend to treat cases like those of the next section.

§3. A weak force condition

In this section will show how dropping (V_4) still leads to the existence of multiple T-periodic solutions of (HS). As was seen in Lemma 1.11, under hypothesis (V_3) - (V_4), if $q \in W_T^{1,2}(\mathbf{R}, \mathbf{R}^n)$ and $I(q) < \infty$, then $q \in \Lambda$. If (V_3) merely holds, this is no longer the case. Functions $q \in E$ can be constructed so that $q(0) = 0$. Thus critical points of I under (V_3) may enter the singularity, i.e. may be "collision orbits". This leads us to the following notion of generalized solution of (HS).

Definition 3.1: A function $q \in C_T(\mathbf{R}, \mathbf{R}^n)$ is a generalized T-periodic solution of (HS) if

  (i) q vanishes on a set, $\mathcal{D}$, of measure 0

  (ii) $q \in C_T^2(\mathbf{R} \backslash \mathcal{D}, \mathbf{R}^n)$

  (iii) q satisfies (HS) on $\mathbf{R} \backslash \mathcal{D}$

  (iv) $q \in E_T$ and $I(q) < \infty$

  (v) $\frac{1}{2} |\dot{q}(t)|^2 + V(q(t)) \equiv constant$ for $t \in \mathbf{R} \backslash \mathcal{D}$.

-15-

Note that energy is conserved for (HS) in each component of $R\backslash \mathcal{D}$. Condition (v) in Definition 3.1 further requires that the constant is the same for each component.

By using an approximation argument based on the results of §1 - 2, we will show that (HS) possesses multiple generalized T-periodic solutions. More precisely we have:

Theorem 3.2: Let V satisfy $(V_1) - (V_3)$. Then for each $T > 0$ (HS) possesses infinitely many distinct generalized T-periodic solutions in E_T.

Proof: For each $\delta > 0$, let $V_\delta(q)$ be a potential satisfying $(V_1) - (V_4)$, $V_\delta = V$ if $|q| > \delta$, and

$$(3.3) \qquad\qquad -V_\delta > -V \quad .$$

Let $k \in \mathbf{N}$ and set

$$(3.4) \qquad\qquad I_{k,\delta}(q) \equiv \int_0^{T/k}\left[\frac{1}{2}\,|\dot{q}|^2 - V_\delta(q)\right]dt \quad .$$

We claim there exist constants α_k, β_k independent of δ such that if $q_{k,\delta} \in W^{1,2}_{T/k} \cap \Lambda$ is the critical point of $I_{k,\delta}$ obtained via Theorem 2.18, then

$$(3.5) \qquad\qquad 0 < \alpha_k < I_{1,\delta}(q_{k,\delta}) < \beta_k \quad .$$

Furthermore α_k can be chosen so that

$$(3.6) \qquad\qquad \alpha_k \to \infty \quad \text{as} \quad k \to \infty \quad .$$

To verify these assertions, let

$$\Lambda_k = \{q \in \Lambda \mid q \text{ is } T/k \text{ periodic}\}$$

and

$$\Gamma_k = \{h \in C(D^{n-2}, \Lambda_k) \mid \deg \tilde{h} \neq 0\} \quad .$$

Choose $h_1 \in \Gamma_1$. Therefore for the associated map $\tilde{h}_1$, there is a constant $\sigma > 0$ such that

$$(3.7) \qquad\qquad \min_{x \in D^{n-2}, t \in [0,T]} |\tilde{h}_1(t)(t)| > \sigma > 0 \quad .$$

Define h_k by

-16-

(3.8)
$$h_k(x)(t) = h_1(x)(kt) \quad .$$

Hence $h_k \in \Gamma_k$ and by (3.8),

(3.9)
$$\min_{x \in D^{n-2}, t \in [0, T/k]} |\tilde{h}_k(x)(t)| > \sigma \quad .$$

Thus if $\delta < \sigma$,

(3.10)
$$V_\delta(h_k(x)(t)) = V(h_k(x)(t)) \quad .$$

Therefore for $\delta < \sigma$,

$$c_k(\delta) \equiv I_{1,\delta}(q_{k,\delta}) = k \inf_{h \in \Gamma_k} \max_{x \in D^{n-2}} I_{k,\delta}(h(x))$$

(3.11)
$$\leqslant k \max_{x \in D^{n-2}} I_{k,\delta}(h_k(x)) = k \max_{x \in D^{n-2}} I_k(h_k(x)) \equiv \beta_k \quad .$$

To get the lower bound in (3.5), let $\varphi_\delta(0) = 0$ and for $s > 0$,

(3.12)
$$\varphi_\delta(s) = \max_{0 < |q| \leqslant s} - (V_\delta(q))^{-1} \quad .$$

Then by (3.3),

(3.13)
$$\varphi_\delta(s) \leqslant \varphi(s)$$

for $s > 0$. Now $c_k(\delta) = k I_{k,\delta}(q_{k,\delta}) \equiv k b_k(\delta)$ and by (2.3),

$$T/k \leqslant b_k(\delta) \varphi_\delta(2(2T_k^{-1} b_k(\delta))^{1/2})$$

or using (3.13),

(3.14)
$$T \leqslant c_k(\delta) \varphi(2(2Tk^{-2} c_k(\delta))^{1/2}) \quad .$$

Define

$$\alpha_k = \inf_{0 < \delta \leqslant \sigma} c_k(\delta) \quad .$$

Passing to a limit in (3.14) shows

(3.15)
$$T \leqslant \alpha_k \varphi(2(2Tk^{-2} \alpha_k)^{1/2}) \quad .$$

Hence the form of φ shows $\alpha_k > 0$ yielding (3.5). Finally if (α_k) were bounded in k along a subsequence, passing to a limit in (3.15) shows $T \leqslant 0$ which is impossible. Therefore (3.6) holds.

Now the existence of generalized T-periodic solutions of (HS) can be established. Fixing k and letting $\delta \to 0$, (3.5) provides upper bounds for $\|\dot{q}_{k,\delta}\|_{L^2}$ and $-\int_0^T V_\delta(q_{k,\delta}(t))dt$. Note that $\xi_{k,\delta} = [q_{k,\delta}]$ must be bounded independently of δ for otherwise $V_\delta(q_{k,\delta}(t))$ and $V'_{k,\delta}(q_{k,\delta}(t)) \to 0$ uniformly for $t \in [0,T]$ and then as in (1.13) - (1.15), $a_k = 0$. Hence $\{q_{k,\delta} \mid \delta < \sigma\}$ are bounded in E. Consequently a subsequence of these functions converges weakly in E and strongly in L^∞ to $q_k \in E$. Moreover

$$(3.16) \qquad\qquad -\int_0^T V(q_k(t))dt \le b_k \quad .$$

Indeed by (3.5), for all $\delta < \sigma$

$$(3.17) \qquad\qquad -\int_0^T V_\delta(q_{k,\delta}(t))dt \le \beta_k \quad .$$

Let $\epsilon > 0$ and $\chi_\epsilon(s) = 0$ if $s < \epsilon$ and $\chi_\epsilon(s) = 1$ if $s \ge \epsilon$. Then (3.17) implies

$$(3.18) \qquad\qquad -\int_0^T \chi_\epsilon(|q_{k,\delta}(t)|)V_\delta(q_{k,\delta}(t))dt \le \beta_k \quad .$$

Since $q_{k,\delta} \to q_k$ in $C_T(\mathbf{R},\mathbf{R}^n)$, (3.18) shows

$$(3.19) \qquad\qquad -\int_0^T \chi_\epsilon(|q_k(t)|)V(q_k(t))dt \le \beta_k \quad .$$

Letting $\epsilon \to 0$ in (3.19) yields (3.17). By (3.17) and (V_3), the set $\mathcal{D}_k$ where q_k vanishes must have measure 0. Let $\tau \in [0,T]\backslash_k$. Then there is an $\epsilon, \omega > 0$ such that $|t-\tau| < \omega$ implies $-V(q_k(t)) \ge \epsilon$. Since $q_{k,\delta} \to q_k$ uniformly in C^2 on $\{|t-\tau| < \omega\}$ (along a subsequence) and $q_{k,\delta}$ satisfies (HS) for V_δ, it readily follows that q_k is a classical solution of (HS) on $\mathbf{R}\backslash\mathcal{D}_k$. Next observe that (v) of Definition 3.1 holds for $q_{k,\delta}$ with $\mathcal{D} = \emptyset$. Hence by passing to a limit we get it for q_k on $\mathbf{R}\backslash\mathcal{D}_k$.

At this point for each $k \in \mathbf{N}$, we have constructed a T/k periodic solution q_k of (HS). Note that if q_k is an equilibrium solution of (HS), $|q_k| \ne 0$ and therefore $q_{k,\delta} \to q_k$ uniformly in $C^2_T(\mathbf{R},\mathbf{R}^n)$ as $\delta \to 0$. Consequently

$$(3.20) \qquad\qquad I_{1,\delta}(q_{k,\delta}) \to I(q_k) \ge \alpha_k \quad .$$

-18-

This observation shows that infinitely many of the functions q_k are distinct. Indeed if a subsequence of the q_k were constant solutions, by (3.20) and (3.6), $I(q_k) \geq \alpha_k \to \infty$ as $k \to \infty$ along this subsequence. On the other hand if there were only finitely many constant solutions, since the period T/k of q_k approaches 0 as $k \to \infty$, infinitely many of the (nonconstant) q_k must be distinct.

Corollary 3.21: If V satisfies $(V_1) - (V_3)$ and (V_5), then for each $T > 0$, (HS) possesses infinitely many distinct nonconstant generalized T-periodic solutions.

Proof: If not, by Theorem 3.2, q_k is a constant solution for all $k \in \mathbf{N}$ and

$$(3.22) \qquad\qquad I(q_k) = -TV(q_k) \geq \alpha_k$$

so $I(q_k) \to \infty$ as $h \to \infty$ by (3.20). But by (V_5), I is bounded on K, a contradiction.

Next consider (2.15)

Definition 3.23: A function $q \in C_T(\mathbf{R}, \mathbf{R}^n)$ is a <u>generalized T-periodic solution of (2.15)</u> if q satisfies (i) - (iv) of Definition 3.1.

With this definition of solution and our understanding of the meaning of $(V_1) - (V_3)$ in the time dependent case, Theorem 2.16 and the argument of Theorem 3.2 show that

Theorem 3.24: If V satisfies $(V_1) - (V_3)$, then (2.15) possesses a generalized T-periodic solutions.

Remark 3.25: As in §2, by replacing (V_1) and (V_3) by (V_1') and (V_3'), we get analogues of Theorems 3.2 and 3.24 for this setting.

-19-

Remark 3.26: The definition of a generalized T-periodic solution allows such
a solution to pass through the origin, indeed for possibly infinitely many
values of $t \in [0,T]$. We suspect that this is not possible for a minimax
solution. An interesting question to pursue is the regularity of the
solutions our minimax procedure produces. Can such a solution actually be a
collision orbit? Alternatively what further conditions on V guarantee
classical solutions?

REFERENCES

[1] Ambrosetti, A. and V. Coti-Zelati, Critical points with lack of compactness and applications to singular dynamical systems, preprint.

[2] Ambrosetti, A. and V. Coti-Zelati, Periodic solutions of singular dynamical systems, in Periodic Solutions of Hamiltonian Systems and Related Topics, P. H. Rabinowitz et. al (eds.), V209, NATO ASI series, Reidel, (1987), 1-10.

[3] Coti-Zelati, V., Morse theory and periodic solutions of Hamiltonian systems, preprint, SISSA, Trieste.

[4] Degiovanni, M., F. Giannoni, and A. Marino, Periodic solutions of dynamical systems with Newtonian type potentials, in Periodic Solutions of Hamiltonian Systems and Related Topics, P. H. Rabinowitz et. al. (eds.), V209, NATO ASI series, Reidel, (1987), 111-115.

[5] Greco, C., Periodic solutions of a class of singular Hamiltonian systems, to appear in Nonlinear Analysis, T.M.A.

[6] Greco, C., Remarks on periodic solutions for some dynamical systems with singularities, in Periodic Solutions of Hamiltonian Systems and Related Topics, P. H. Rabinowitz et. al. (eds.), V209, NATO ASI series, Reidel, (1987, 169-173.

[7] Gordon, W. B., Conservative dynamical systems involving strong forces, Trans. A.M.S. 204, (1975), 113-135.

[8] McGehee, R., Double collisions for a classical particle system with nongravitational interactions, Comment. Math. Helvetici, 56, (1981), 524-557.

[9] Rabinowitz, P. H., Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. in Math. #65, Amer. Math. Soc., Providence, RI, 1986.

AB/PHR/jvs

END
DATE
FILMED
7-88
DTIC